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A new recursion relation of CFP for the system of identical bosons

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Abstract. A new recursion relation of coefficients of fractional parentage (CFP) with seniority presented here provides an efficient algorithm for computation. As a result of the generalised Wigner-Eckart theorem of semisimple Lie groups, the CFP are factorised into isoscalar factors (ISF) of the symmetry group of n bosons.

The evaluations are done in the scheme of second quantisation. An analytic formula of the multiplicity of an irreducible representation (IR) of O(3) in an IR of O(N) is presented in this paper. A complete label (including seniority) of the states for the system of bosons, each with angular momentum l, is also presented.

1. Introduction

The main difficulty in constructing the state vector of an identical particle system possessing spherical symmetry is that the state vector possesses well defined permutational symmetry and total angular momentum also. It is known that the CFP method [1] is very powerful for solving the problem; in particular the state vectors expressed in CFP make the calculation of matrix elements of a one-body tensor operator and a two-body scalar operator simple. So the evaluation of the CFP is quite important in the field of microphysics.

There are several procedures for the evaluation of CFP [2] in iterative form which have recently been improved in numerical computation [3] and in the algorithm for carrying out CFP calculations [4, 5]. In this paper a new recursion relation with well defined seniority for the CFP of the system of bosons, each with angular momentum l, is presented. A kind of generalised irreducible tensor operator of a semisimple Lie group is defined here. So, using the generalised Wigner-Eckart theorem, CFP are factorised into a product of ISF for the special unitary and orthogonal groups. Then the evaluation of the CFP is reduced to evaluating ISF for the group chain $O(N)\otimes O(N)\supset O(N)\supset O(3)$. Thus the computing becomes efficient, especially for relatively large number of bosons. This paper also presents methods for analytically computing the multiplicity occurring in the reduction of O(N) to O(3) and for completely labelling the state vectors.

The generalised definition of CFP in second quantisation is developed in § 2. The relation between CFP and ISF for the boson system classified according to group chain $U(N)\supset O(N)\supset O(3)$ is given in § 3. Section 4 is devoted to the deduction of the recursion relation of CFP.

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2. CFP method

For a system of n bosons, each with angular momentum l, the symmetrised state vector with total angular momentum L is

$$|l^n \alpha \ L \ M\rangle \equiv |n \ \alpha \ L \ M\rangle$$

where α symbolises additional quantum numbers required to completely label the state. Using coupling of angular momenta, we have

$$\begin{aligned}|n-1 \ \alpha' \ L' \ l \ L \ M\rangle &= \{|n-1 \ \alpha' \ L'\rangle|l\rangle\}^L_M \\ &= \sum_m \langle L' \ M' \ l \ m|L \ M\rangle|n-1 \ \alpha' \ L' \ M'\rangle|l \ m\rangle\end{aligned}$$

where $\langle L' \ M' \ l \ m|L \ M\rangle$ are the CG coefficients. In this form the state $|n-1 \ \alpha' \ L' \ l \ L \ M\rangle$ has not been symmetrised between nth particle and the first $n-1$ particles. The symmetrised state $|n\alpha LM\rangle$ is connected with the unsymmetrised one $|n-1 \ \alpha' \ L' \ l \ L \ M\rangle$ by a linear transformation, i.e.

$$|n \ \alpha \ L \ M\rangle = \sum_{\alpha' L'} \langle n-1 \ \alpha' \ L' \ l \ L|\}n \ \alpha \ L\rangle|n-1 \ \alpha' \ L' \ l \ L \ M\rangle.$$

The elements of the transformation $\langle n-1 \ \alpha' \ L' \ l|\}n \ \alpha \ L\rangle$ are called single-particle CFP and the states $|n-1 \ \alpha' \ L' \ l \ L \ M\rangle$ are called parentage states. A reasonable choice of phase makes the CFP real. Notice that this transformation is not unitary. So in the notation of CFP, the symbol $|\}$ is used instead of $|$, indicating the absence of an inverse transformation because all but symmetric representations are suppressed in the coupling of a particle to a symmetric $(n-1)$-particle state.

The reduced matrix elements of a one-body tensor operator, $T^k_q = \Sigma_i \, t^k_q(i)$, may also with advantage be expressed as

$$\begin{aligned}&\langle n \ \tilde{\alpha} \ \tilde{L}\|T^k\|n \ \alpha \ L\rangle \\ &\quad = n\langle l\|t^k\|l\rangle \sum_{\alpha' L'} \langle n-1 \ \alpha' \ L' \ l \ L|\}n \ \alpha \ L\rangle\langle n-1 \ \alpha' \ L' \ l \ \tilde{L}|\}n \ \tilde{\alpha} \ \tilde{L}\rangle \\ &\quad \times \langle (kl) \ l \ L' \ \tilde{L}|k \ (lL') \ L \ \tilde{L}\rangle\end{aligned} \tag{2.1}$$

where $\langle (kl)l \ L' \ \tilde{L}|k \ (l \ L') \ L \ \tilde{L}\rangle$ are the normalised Racah coefficients. In second quantisation the creation operator $b^\dagger_{lm}$ and annihilation operator b_{lm} of a boson with angular momentum l and its z component $l_z = m$ satisfy the following commutation relations:

$$[b_{lm}, b_{lm'}] = [b^\dagger_{lm}, b^\dagger_{lm'}] = 0$$

$$[b_{lm}, b^\dagger_{lm'}] = \delta_{mm'}.$$

Since all the states $b^\dagger_{lm_1} b^\dagger_{lm_2} \ldots |0\rangle$ are symmetrised, there is no parantage state in the second quantisation scheme. But the reduced matrix elements can be expressed in terms of $b^\dagger_{lm}$, namely

$$\begin{aligned}&\langle n \, \tilde{\alpha} \, \tilde{L}|T^k|n \ \alpha \ L\rangle \\ &\quad = \langle l\|t^k\|l\rangle \sum_{\alpha' L'} \langle n \ \tilde{\alpha} \ \tilde{L}\|b^\dagger_l\|n-1 \ \alpha' \ L'\rangle\langle n \ \alpha \ L\|b^\dagger_l\|n-1 \ \alpha' \ L'\rangle \\ &\quad \times \langle (kl) \ l \ L' \ \tilde{L}|k \ (lL') \ L \ \tilde{L}\rangle.\end{aligned} \tag{2.2}$$

Comparing (2.2) with (2.1), one gets a formal definition of CFP in second quantisation

$$\langle n-1 \;\; \alpha' \;\; L' \;\; l \;\; L|\}n \;\; \alpha \;\; L\rangle = \sqrt{\frac{1}{n}}\langle n \;\; \alpha \;\; L\|b_l^\dagger\|n-1 \;\; \alpha' \;\; L'\rangle. \tag{2.3}$$

The two-body scalar operator $V=\frac{1}{2}\Sigma_{ij}\, v_{ij}$ can now be written as

$$V=\sum_{\text{all } m}\langle m_1 \;\; m_2|v|m_3 \;\; m_4\rangle b_{lm_1}^\dagger b_{lm_2}^\dagger b_{lm_4} b_{lm_3}.$$

By recoupling of four angular momenta l we may bring V into an equivalent form in terms of a one-body tensor operator, namely

$$V=\sum_k \varepsilon_k\{\hat{n}\delta_{k0}-\sqrt{2k+1}(B(ll)^k\cdot B(ll)^k)_0^0\} \tag{2.4}$$

where

$$\varepsilon_k=\sum_{k'=\text{even}}\langle l^2 \;\; k|v|l^2 \;\; k\rangle(2k+1)\begin{Bmatrix} l & l & k' \\ l & l & k \end{Bmatrix}$$

$$\hat{n}=\sum_m b_{lm}^\dagger b_{lm}$$

$$B(ll)_q^k=(b_l^\dagger \tilde{b}_l)_q^k=\sum_m \langle l \;\; m \;\; l \;\; m'|kq\rangle b_{lm}^\dagger \tilde{b}_{lm'}$$

$$\tilde{b}_{lm}=(-1)^{l+m}b_{l-m}$$

and $\{{}^{l\,l\,k'}_{l\,l\,k}\}$ is the 6-j symbol. This form gives us a clue to computing the matrix elements of V by directly using single-particle CFP without the definition of two-particle CFP, even though one can formally define them by the reduced matrix elements of V in accordance with the definition (2.3). It is necessary to mention that one must take care of the additional one-body term n when dealing with V in the form (2.4).

3. Factorisation of CFP

The symmetry group for the number-preserved system of bosons, each with angular momentum l, is the unitary group $U(N)=U(2l+1)$, whose generators are $B(ll)_q^k$ defined above. Furthermore, the state vectors of n bosons of this kind can be classified according to the group chain $U(N)\supset O(N)\supset O(3)$, i.e.

$$|[n] \;\; (\sigma) \;\; \alpha \;\; L \;\; M\rangle\equiv|n \;\; \sigma \;\; \alpha \;\; L \;\; M\rangle \tag{3.1}$$

where σ is the seniority number, labelling IR of $O(N)$ and relative to the boson pair creation operator—the invariant of $O(N)$

$$P_l^\dagger=\sqrt{\frac{2l+1}{2}}(b_l^\dagger b_l^\dagger)_0^0 \tag{3.2}$$

$[n]$ labels fully symmetric IR of $U(N)$.

All generators for these groups are presented in table 1.

The generalised irreducible tensor operator $T(\Gamma,\gamma)$ of rank Γ of semisimple Lie group G, a concept extended from O(3), is defined as follows [6, 7]:

$$[X_i, T(\Gamma,\gamma)]=\sum_{\gamma'}\langle\Gamma \;\; \gamma'|X_i|\Gamma \;\; \gamma\rangle T(\Gamma,\gamma') \tag{3.3}$$

Table 1. The generators and Casimir operators for the group chain $U(N) \supset O(N) \supset O(3)$ in the boson realisation.

Group	Generators	Casimir operator
$U(N)$	$B(ll)^k_q = (b^\dagger_l \tilde{b}_l)^k_q$ $k = 2l, 2l-1, \ldots, 0$	$C_1 = \hat{n} = \sqrt{2l+1}\, B(ll)^0_0$ $C_2 = \sum_k \sqrt{2k+1}(B(ll)^k B(ll)^k)^0_0$ $= \hat{n}(\hat{n} + N - 1)$
$O(N)$	$O(ll)^k_q = 2B(ll)^k_q$ $k = 2l-1, 2l-3, \ldots, 1$	$C_2 = -\frac{1}{2}\sum_k \sqrt{2k+1}(O(ll)^k O(ll)^k)^0_0$ $= \hat{n}(\hat{n} + N - 2) - 2P^\dagger_l P_l$
$O(3)$	$L_q = \left(\frac{l(l+1)(2l+1)}{3}\right)^{1/2} B(ll)^1_q$	$C_2 = LL = L^2$

where X_i are the generators of G; Γ labels IR of G and γ labels its basis. Then the reduced matrix elements of $T(\Gamma, \gamma)$ can be defined by the generalised Wigner–Eckart theorem, namely

$$\langle \tilde{\Gamma} \ \tilde{\gamma} | T \ (\bar{\Gamma}\bar{\gamma}) | \Gamma \ \gamma \rangle = \left[\begin{array}{cc|c} \bar{\Gamma} & \Gamma & \tilde{\Gamma} \\ \bar{\gamma} & \gamma & \tilde{\gamma} \end{array}\right] \langle \tilde{\Gamma} \| T(\bar{\Gamma}) \| \Gamma \rangle \tag{3.4}$$

where $\langle \tilde{\Gamma} \| T(\bar{\Gamma}) \| \Gamma \rangle$ symbolise the reduced matrix elements of G.

It is straightforward to prove that the operators

$$b^\dagger_{lm} = b^\dagger([1](1)lm) \equiv b^\dagger(11lm)$$

form an irreducible tensor operator of rank [1] of $U(N)$ satisfying

$$\begin{aligned} &[B(ll)^k_q, b^\dagger(11lm)] \\ &\quad = \sum_{m'} \langle 1 \ 1 \ l \ m' | B(ll)^k_q | 1 \ 1 \ l \ m \rangle b^\dagger(11lm') \end{aligned} \tag{3.5}$$

where $|11lm\rangle = |[1](1)lm\rangle = b^\dagger_{lm}|0\rangle$ are the basis of IR [1] of $U(N)$, built up by $b^\dagger_{lm}$. Equation (3.5) also shows that $\{b^\dagger_{lm}\}$ is an irreducible tensor operator of rank (1) of $O(N)$ and rank l of O(3). One can say that $\{b^\dagger_{lm}\}$ is classified according to the group chain $U(N) \supset O(N) \supset O(3)$. Then, making use of (3.4), we have

$$\begin{aligned} &\langle n \ \sigma \ \alpha \ L \| b^\dagger(11l) \| n-1 \ \sigma' \ \alpha' \ L' \rangle \\ &\quad = \langle n \| b^\dagger \| n-1 \rangle \left[\begin{array}{cc|c} [1] & [n-1] & [n] \\ (1) & (\sigma') & (\sigma) \end{array}\right] \left[\begin{array}{cc|c} (1) & (\sigma') & (\sigma) \\ l & \alpha' L' & \alpha L \end{array}\right] \end{aligned} \tag{3.6}$$

where

$$\left[\begin{array}{cc|c} [1] & [n-1] & [n] \\ (1) & (\sigma') & (\sigma) \end{array}\right] \qquad \left[\begin{array}{cc|c} (1) & (\sigma') & (\sigma) \\ l & \alpha' L' & \alpha L \end{array}\right]$$

are ISF of $U(N)$ and $O(N)$ respectively. Since $\langle n \| b^\dagger \| n-1 \rangle$, the $U(N)$ reduced matrix element, is independent of σ and L, a special representation $n = \sigma$, $L = nl$, $\sigma' = n - 1 = \sigma - 1$, $L' = (n-1)l$ may be selected which makes the ISF unity. Hence

$$\langle n \| b^\dagger \| n-1 \rangle = \sqrt{n}.$$

The factorised CFP are obtained by substituting (3.6) for (2.3), i.e.

$$\langle n-1\ \sigma'\ \alpha'\ L'\ l\ L|\}n\ \sigma\ \alpha\ L\rangle = \begin{bmatrix}[1] & [n-1] & [n] \\ (1) & (\sigma') & (\sigma)\end{bmatrix}\begin{bmatrix}(1) & (\sigma') & (\sigma) \\ l & \alpha' L' & \alpha L\end{bmatrix}. \tag{3.7}$$

This is the relation between CFP and ISF. The right-hand side of (3.7) represents the permutation symmetry $[n-1]\otimes[1]\supset[n]$, but clearly includes a factor of the coupling of angular momentum $L\otimes L'\supset L$. So (3.7) clarifies the physical meaning of CFP.

The state vectors (3.1) can be expressed in terms of the invariant $P_l^\dagger$ of O(N) defined in (3.2), i.e.

$$|n\ \sigma\ \alpha\ L\rangle = CP_l^{\dagger\rho}|\sigma\ \sigma\ \alpha\ L\rangle \tag{3.8}$$

where $|\sigma\,\sigma\,\alpha\,L\rangle$ satisfy the following equation:

$$P_l|\sigma\ \sigma\ \alpha\ L\rangle = 0 \tag{3.9}$$

and $\rho = \frac{1}{2}(n-\sigma)$ is the number of boson pairs. Using the commutation relation

$$[P_l, P_l^{\dagger\rho}] = \rho(2\hat{n}+N-2\rho+2)P_l^{\dagger(\rho-1)} \tag{3.10}$$

we have the normalised constant

$$C = \left(\frac{(2\sigma+N-2)!!}{\rho!(2\sigma+N+2\rho-2)!!}\right)^{1/2}.$$

By means of the commutator

$$[P_l^{\rho}, b_{lm}^\dagger] = \sqrt{2}\rho P_l^{(\rho-1)}\tilde{b}_{lm} \tag{3.11}$$

the reduced matrix elements of $b_{lm}^\dagger$ may then be computed; they are

$$\langle n\ \sigma\ \alpha\ L\|b_l^\dagger\|n-1\ \ \sigma-1\ \ \alpha'\ L'\rangle$$
$$= \sqrt{n}\left(\frac{n+\sigma+N-2}{n(2\sigma+N-2)}\right)^{1/2}\langle\sigma\ \alpha\ L\|b_l^\dagger\|\sigma-1\ \ \alpha'\ L'\rangle \tag{3.12}$$

and

$$\langle n\ \sigma\ \alpha\ L\|b_l^\dagger\|n-1\ \ \sigma+1\ \ \alpha'\ L'\rangle$$
$$= \sqrt{n}\left(\frac{n-\sigma}{n(2\sigma+N)}\right)^{1/2}\langle\sigma\ \alpha\ L\|\tilde{b}_l\|\sigma+1\ \ \alpha'\ L'\rangle. \tag{3.13}$$

These two results clearly show that the evaluation of irreducible matrix elements of $b_{lm}^\dagger$ for $n\geqslant\sigma$ is reduced to that for $n=\sigma$.

Comparison between (3.12), (3.13) and (3.6) and use of the orthogonality of ISF give the algebraic expression of $\left[\begin{smallmatrix}[1] & [n-1] & [n_1n_2] \\ (1) & (\sigma') & (\sigma_1\sigma_2)\end{smallmatrix}\right]$ in table 2. Then we have

$$\begin{bmatrix}(1) & (\sigma-1) & (\sigma) \\ l & \alpha' L' & \alpha L\end{bmatrix} = \frac{\langle\sigma\alpha L\|b_l^\dagger\|\sigma-1\ \ \alpha'\ L'\rangle}{\sqrt{\sigma}} \tag{3.14}$$

Table 2. ISF of U(N).

σ	$[n_1n_2]=[n]$ $(\sigma_1\sigma_2)=(\sigma)$	$[n_1n_2]=[n-1\ \ 1]$ $(\sigma_1\sigma_2)=(\sigma)$	$[n_1n_2][n-1\ \ 1]$ $(\sigma_1\sigma_2)=(\sigma+1\ \ 1)$	$[n_1n_2]=[n-1\ \ 1]$ $(\sigma_1\sigma_2)=(\sigma-1\ \ 1)$
$\sigma-1$	$\left(\frac{\sigma(n+\sigma+N-2)}{n(2\sigma+N-2)}\right)^{1/2}$	$\left(\frac{(n-\sigma)(\sigma+N-2)}{n(2\sigma+N-2)}\right)^{1/2}$	0	1
$\sigma+1$	$\left(\frac{(n-\sigma)(\sigma+N-2)}{n(2\sigma+N-2)}\right)^{1/2}$	$-\left(\frac{\sigma(n+\sigma+N-2)}{n(2\sigma+N-2)}\right)^{1/2}$	1	0

and

$$\begin{bmatrix} (1) & (\sigma+1) & (\sigma) \\ l & \alpha' L' & \alpha L \end{bmatrix} = \left(\frac{2\sigma+N-2}{(\sigma+N-2)(2\sigma+N)} \right)^{1/2} \langle \sigma \ \alpha \ L \| \tilde{b}_l \| \sigma+1 \ \alpha' \ L' \rangle. \tag{3.15}$$

Making use of the relation

$$\langle \sigma-1 \ \alpha' \ L' \| \tilde{b}_l \| \sigma \ \alpha \ L \rangle = (-1)^{l+L+L'} \left(\frac{2L+1}{2L'+1} \right) \langle \sigma \ \alpha \ L \| b_l^\dagger \| \sigma-1 \ \alpha' \ L' \rangle \tag{3.16}$$

we may obtain the well known reciprocal rule of ISF for $O(N) \otimes O(N) \supset O(N) \supset O(3)$

$$\begin{bmatrix} (1) & (\sigma) & (\sigma-1) \\ l & \alpha L & \alpha' L' \end{bmatrix} = (-1)^{l+L+L'} \left(\frac{\sigma(2\sigma+N-4)(2L+1)}{(\sigma+N-3)(2\sigma+N-2)(2L'+1)} \right)^{1/2} \times \begin{bmatrix} (1) & (\sigma-1) & (\sigma) \\ l & \alpha' L' & \alpha L \end{bmatrix}. \tag{3.17}$$

To sum up, the evaluation of CFP defined in (2.3) are reduced to that of $\langle \sigma \, \alpha \, L \| b_l^\dagger \| \sigma-1 \ \alpha' \ L' \rangle$.

4. The recursion relation of ISF for $O(N) \otimes O(N) \supset O(N) \supset O(3)$

Since the reduction on $O(N) \supset O(3)$ is generally not simple, the first step to evaluate ISF is to find the multiplicity of L in (σ) and choose a practical α in $|n \, \sigma \, \alpha \, L \, M\rangle$. We may introduce a quantity $\alpha(\sigma L)$ to symbolise the multiplicity of L in (σ). It is easy to prove that for $L = nl - \xi$

$$\alpha(\sigma \ L) = N(\sigma \ \xi) - N(\sigma-2 \ \xi) + N(\sigma-2 \ \xi-1) - N(\sigma \ \xi-1) \tag{4.1}$$

where $N(n\xi)$ is the number of states with quantum number $M = nl - \xi > 0$ and it is equal to the number of partitions $[\xi_1 \xi_2 \ldots \xi_k]$ of integer $\xi \geqslant 0$ under the following conditions:

$$\xi = \sum_i \xi_i \qquad \xi_i \text{ integer}$$

$$\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_k \geqslant 0 \qquad k \leqslant n$$

$$\xi_1 \leqslant 2l.$$

Consider the fact that the reduction of $O(3) \otimes O(3) \supset O(3)$ is simple, we see that the state with L in a multiplicity comes from different constituent angular momentum L_0. Suppose we start from the state $|\sigma-1 \ \alpha'_0 \ L'_0 \ M'_0\rangle$, which has well defined $\sigma-1$, α'_0, L'_0, M'_0. Making use of CG coefficients we may define a state with well defined L and M

$$\begin{aligned} |\psi_{LM}\rangle &= \{ b_l^\dagger | \sigma-1 \ \alpha'_0 \ L'_0 \rangle \}^L_M \\ &= \sum_m \langle l \ m \ L'_0 \ M'_0 | L \ M \rangle b^\dagger_{lm} | \sigma-1 \ \alpha'_0 \ M'_0 \rangle. \end{aligned}$$

Notice that this state has no well defined seniority because if one boson is added, the seniority changes by ±1. But this state may be given in terms of states with σ and $\sigma-2$ by linear composition, namely

$$|\psi_{LM}\rangle = a|\sigma \ (\alpha'_0 \ L'_0) \ L \ M\rangle + \sum_{\alpha''} b_{\alpha''} |\sigma \ \sigma-2 \ \alpha'' \ L \ M\rangle \tag{4.2}$$

where in $|\sigma\,(\alpha_0' L_0')\; L\, M\rangle$, $(\alpha_0' L_0')$ is used instead of α. Obviously, α_0' in $|\sigma-1\;\alpha_0'\;L_0'\;M_0'\rangle$ may be substituted for the constituent one $(\alpha_0''\;L_0'')$ of L_0'. So this kind of label is unique and states $|\sigma\,(\alpha_0'\;L_0')\;L\,M\rangle$ are generally supercomplete. Nevertheless, it is easy to select a complete set (the number of linear independent states equal to multiplicity $\alpha(\sigma L)$) from them by an orthogonalising program. Therefore, it is feasible to use $(\alpha_0'\;L_0')$ to completely label the basis although there is no operator corresponding to $(\alpha_0'\;L_0')$ being found. After a tedious derivation we finally get the recursion relation of ISF

$$\left[\begin{array}{cc|c} (1) & (\sigma-1) & (\sigma) \\ l & \alpha' L' & (\alpha_0' L_0') L \end{array}\right] = \frac{R(\alpha_0' L_0' \alpha' L' L)}{\sqrt{\sigma R(\alpha_0' L_0' \alpha_0' L_0' L)}} \tag{4.3}$$

where

$$\begin{aligned} R(\alpha_0' L_0' \alpha' L' L) &= \delta(\alpha_0' \alpha')\delta(L_0' L') + (\sigma-1) \sum_{\alpha'' L''} (-1)^{L_0'+L'} \sqrt{(2L_0'+1)(2L'+1)} \\ &\times \left(\begin{Bmatrix} l & L'' & L_0' \\ l & L & L' \end{Bmatrix} - \frac{2\delta(L'' L)}{(2\sigma+N-4)(2L+1)} \right) \\ &\times \left[\begin{array}{cc|c} (1) & (\sigma-2) & (\sigma-1) \\ l & \alpha'' L'' & \alpha_0' L_0' \end{array}\right] \left[\begin{array}{cc|c} (1) & (\sigma-2) & (\sigma-1) \\ l & \alpha'' L'' & \alpha' L' \end{array}\right]. \end{aligned}$$

Specifically $a^2 = R(\alpha_0' L_0' \alpha_0' L_0' L)$ and naturally the iterative from (4.3) is independent of all $b_{\alpha''}$ in (4.2). Thus the ISF (hence the CFP) with any σ are reduced to that with $\sigma = 3$ and built up by use of (4.3). That this recursion relation depends on σ, not on n, avoids a lot of repetitive computation relative to identical (σ) in differential $[n]$. Furthermore, the dimension of (σ) is smaller than that of $[n]$ including (σ). Surely this new recursion relation (4.3) including seniority provides a faster mechanism for computation of CFP than others without σ.

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